WHAT EVERY YOUNG MATHLETE SHOULD KNOW

I. VOCABULARY AND LANGUAGE

The following explains, defines, or lists some of the words that may be used in Olympiad problems. To be accepted, an answer must be consistent with both this document and the wording of the problem.

1. **BASIC TERMS**

Sum, difference, product, quotient, remainder, ratio, square of a number (also, perfect square), factors of a number. The ***value*** of a number is the simplest name for that number. "**Or**" is inclusive: "*a* or *b*" means "*a* or *b* or both."

DIVISION M: Square root of a number, cube of a number (also, perfect cube).

2. **READING SUMS**

An ellipsis (…) means "and so forth":

Read "1 + 2 + 3 + …" as "one plus two plus three and so forth (without end)".

Read "1 + 2 + 3 + … + 10" as "one plus two plus three and so forth *up to ten.*"

3. **STANDARD FORM OF A NUMBER**

The **standard form of a number** refers to the form in which we usually write numbers (also called Hindu-Arabic numerals or positional notation).

A **digit** is any one of the ten numerals 0,1,2,3,4,5,6,7,8,9. Combinations of digits are assigned place values in order to write all numbers. A number may be described by the number of digits it contains: 358 is a three-digit number. The "**lead-digit**" (leftmost digit) of a number is not counted as a digit if it is 0: 0358 is a *three-digit* number. **Terminal zeros** of a number are the zeros to the right of the last nonzero digit: 30,500 has two terminal zeros because to the right of the digit 5 there are two zeros.

4. **SETS OF NUMBERS**

Counting Numbers = {1, 2, 3, …}.

Whole Numbers = {0, 1, 2, 3, …}

DIVISION M: **Integers** = {…, –3, –2, –1, 0, +1, +2, +3, …}. The terms **positive, negative, nonnegative,** and **nonpositive numbers** will appear only in Division M problems.

Consecutive numbers are counting numbers that differ by 1, such as 83, 84, 85, 86, and 87.

Consecutive even numbers are multiples of 2 that differ by 2, such as 36, 38, 40, and 42.

Consecutive odd numbers are nonmultiples of 2 that differ by 2, such as 57, 59, 61, and 63.

5. **MULTIPLES, DIVISIBILITY AND FACTORS**

The product of two whole numbers is called a **multiple** of each of the whole numbers. Zero is considered a multiple of every whole number. *Example: Multiples of 6 = {0,6,12,18,24,30,…}.*

> *Note: Many but not all authorities expand the definition of multiple to include all integers. To them, –24 is a multiple of 6. For Olympiad problems, no multiples will be negative.*

A whole number ***a*** is said to be **divisible by** a counting number ***b*** if *b* divides *a* with zero remainder. In such instances: (1) their quotient is also a whole number, (2) *b* is called a **factor** of *a*, and (3) *a* is called a **multiple** of *b*.

6. **NUMBER THEORY**

a. A **prime number** (also, **prime**) is a counting number which has exactly two different factors, namely the number itself and the number 1. *Examples: 2, 3, 5, 7, 11, 13, …*

b. A **composite number** is a counting number which has at least three different factors, namely the number itself, the number 1, and at least one other factor. *Examples: 4, 6, 8, 9, 10, 12, …*

c. The number 1 is neither prime nor composite since it has exactly one factor, namely the number itself. Thus, there are 3 separate categories of counting numbers: prime, composite, and the number 1.

d. A number is **factored completely** when it is expressed as a product of only prime numbers.
Example: $144 = 2\times2\times2\times2\times3\times3$. It may also be written as $144 = 2^4\times3^2$.

e. The **Greatest Common Factor (GCF)** of two counting numbers is the largest counting number that divides each of the two given numbers with zero remainder. *Example: GCF(12,18) = 6.*

f. If the GCF of two numbers is 1, then we say the numbers are **relatively prime** or **co-prime**.

g. The **Least Common Multiple (LCM)** of two counting numbers is the smallest counting number that each of the given numbers divides with zero remainder. *Example: LCM(12,18) = 36.*

h. **Order of operations**. When computing the value of expressions involving two or more operations, the following priorities must be observed from left to right:
1) do operations in parentheses, braces, or brackets first, working from the inside out,
2) do multiplication and division from left to right, and then
3) do addition and subtraction from left to right.

Example:
$$\begin{aligned} &3 + 4 \times 5 - 8 \div (9 - 7)\\ &= 3 + 4 \times 5 - 8 \div 2\\ &= 3 + 20 - 4\\ &= 19\end{aligned}$$

7. FRACTIONS

a. A **common (or simple) fraction** is a fraction of the form $\frac{a}{b}$ where a is a whole number and b is a counting number. One meaning is $a \div b$.

b. A **unit fraction** is a common fraction with numerator 1.

c. A **proper fraction** is a common fraction in which $a < b$. Its value is more than 0 and less than 1.

d. An **improper fraction** is a common fraction in which $a \geq b$. Its value is 1 or greater than 1. A fraction whose denominator is 1 is equivalent to an integer.

e. A **complex fraction** is a fraction whose numerator or denominator contains a fraction. They can be simplified by dividing the numerator by the denominator.

Examples: $\dfrac{\frac{2}{3}}{5}$, $\dfrac{7}{\frac{3}{8}}$, $\dfrac{\frac{2}{3}}{\frac{5}{7}}$, $\dfrac{3-\frac{1}{3}}{3+\frac{1}{3}}$

f. The fraction $\frac{a}{b}$ is **simplified** ("in lowest terms") if a and b have no common factor other than 1 [**GCF(*a*,*b*) = 1**].

g. A **decimal** or **decimal fraction** is a fraction whose denominator is a power of ten. The decimal is written using decimal point notation. *Examples:* $\frac{7}{10}$ *= .7; .36, .005, 1.4*

h. **DIVISION M:** A **percent** or **percent fraction** is a fraction whose denominator is 100, which is represented by the percent sign. *Examples:* $\frac{45}{100}$ *= 45%; 8%, 125%, 0.3%*

8. STATISTICS AND PROBABILITY

The **average** (**arithmetic mean**) of a set of N numbers is the sum of all N numbers divided by N. The **mode** of a set of numbers is the number listed most often. The **median** of an *ordered set* of numbers is the middle number if N is odd, or the mean of the two middle numbers if N is even.

The **probability** of an event is a value between 0 and 1 inclusive that expresses how likely an event is to occur. It is often found by dividing the number of times an event *does* occur by the total number of times the event *can* possibly occur. *Example*: The probability of rolling an odd number on a standard die is $\frac{3}{6}$ or $\frac{1}{2}$. Either $\frac{3}{6}$ or $\frac{1}{2}$ will be accepted as a correct probability.

9. **GEOMETRY**

a. **Angles**: degree-measure, vertex, congruent; acute, right, obtuse, straight, reflex.
b. **Congruent segments** are two **line segments** of equal length.
c. **Polygons**, **circles**, and **solids**:

Parts: side, angle, vertex, diagonal; interior region, exterior region; diameter, radius, chord.
Triangles: acute, right, obtuse; scalene, isosceles, equilateral.
Note: an equilateral triangle is isosceles with all three sides congruent.
Quadrilaterals: parallelogram, rectangle, square, trapezoid, rhombus.
Note: ***a square is one type of rectangle*** *with all four sides congruent. It is also a rhombus with all four angles congruent.*
Others: cube, rectangular solid; pentagon, hexagon, octagon, decagon, dodecagon, icosagon.
Perimeter: the number of unit lengths in the boundary of a plane figure.
Area: the number of unit squares contained in the interior of a region.
Circumference: the perimeter of a circular region.
Congruent figures: two or more plane figures all of whose corresponding pairs of sides and angles are congruent.
Similar figures: two or more plane figures whose size may be different but whose shape is the same. *Note: all squares are similar; all circles are similar.*

d. **DIVISION M:** Geometric Solids: Right Circular Cylinder, face, edge.
Volume: the number of unit cubes contained in the interior of a solid.
Surface Area: the sum of the areas of all the faces of a geometric solid.

II. SKILLS

1. **COMPUTATION**

The tools of arithmetic are needed for problem-solving. Competency in the basic operations on whole numbers, fractions, and decimals is essential for success in problem solving at all levels.

DIVISION M: Competency in basic operations on signed numbers should be developed.

2. **ANSWERS**

Unless otherwise specified in a problem, equivalent numbers or expressions should be accepted. For example, 3½, $^7/_2$, and 3.5 are equivalent.

Units of measure are rarely required in answers but if given in an answer, they must be correct. More generally, an answer in which any part is incorrect is not acceptable. To avoid the denial of credit students should be careful to include *only* required information. While an answer that differs from the official one can be appealed, credit can be granted only if the wording of the problem allows for an alternate interpretation or if it is flawed so that no answer satisfies all conditions of the problem.

Measures of area are usually written as square units, sq. units, or units^2. For example, square centimeters may be abbreviated as sq cm, or cm × cm, or cm^2. In **DIVISION M**, cubic measures are treated in a like manner.

After reading a problem, a wise procedure is to indicate the nature of the answer at the bottom of a worksheet before starting the work necessary for solution. Examples: "*A = ___, B = ___*"; "*The largest number is* __". Another worthwhile device in practice sessions is to require the student to write the answer in a simple declarative sentence using the wording of the question itself. Example: "*The average speed is 54 miles per hour*." This device usually causes the student to reread the problem.

3. **MEASUREMENT**

The student should be familiar with units of measurement for time, length, area, and weight (and for **DIVISION M**, volume) in English and metric systems. Within a system of measurement, the student should be able to convert from one unit to another.

III. SOME USEFUL THEOREMS

1. If a number is divisible by 2^n, then the number formed by the last n digits of the given number is also divisible by 2^n; and conversely.

   Example: 7,292,536 is divisible by 2 (or 2^1) because 6 is divisible by 2.

   Example: 7,292,536 is divisible by 4 (or 2^2) because 36 is divisible by 4.

   Example: 7,292,536 is divisible by 8 (or 2^3) because 536 is divisible by 8.

2. If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.
   If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.

   Example: 658,773 is divisible by 9 because 6+5+8+7+7+3 = 36 which is a multiple of 9.

   Example: 323,745 is divisible by 3 because 3+2+3+7+4+5 = 24 which is a multiple of 3.

3. A number is divisible by 5 if its units digit is 5 or 0.

4. A number is divisible by 11 if the difference between the sum of the odd-place digits and the sum of the even-place digits is 0 or a multiple of 11.

   Example: 90,728 is divisible by 11 because (9+7+8) – (0+2) = **22**, which is a multiple of 11.

5. If A and B are natural numbers, then:

   (i) GCF(A,B) × LCM(A,B) = A × B.

   (ii) LCM(A,B) = (A × B) ÷ GCF(A,B).

   (iii) GCF(A,B) = (A × B) ÷ LCM(A,B).

   Example: If A = 9 and B = 12: GCF(9,12) = 3, LCM(9,12) = 36, A × B = 9 × 12 =108.
   Then: (i) 3 x 36 = 108; (ii) 108 ÷ 3 = 36; (iii) 108 ÷ 36 = 3.

6. If p represents a prime number, then p^n has n+1 factors. Example: 2×2×2×2×2 = 2^5 has 6 factors which are 1, 2, 2×2, 2×2×2, 2×2×2×2, 2×2×2×2×2. In **exponential form**, the factors are: 1, 2, 2^2, 2^3, 2^4, and 2^5. In **standard form**, the factors are: 1, 2, 4, 8, 16, and 32. Notice that the factors of 2^5 include both 1 and 2^5.

   Problem: *how many factors does 72 have?* 72 = 2×2×2×3×3 = 2^3×3^2. Since 2^3 has 4 factors and 3^2 has 3 factors, 72 has 4×3 = 12 factors. The factors may be obtained by multiplying any one of the factors of 2^3 by any one of the factors of 3^2: (1, 2, 2^2, 2^3) × (1, 3, 3^2). Written in order, the 12 factors are: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.

7. Any number divisible by x is divisible by every factor of x and should be checked for the highest power of each prime factor. For example, test multiples of 72 for divisibility by 2^3 and 3^2.

Thorough discussions of these and many other useful topics may be found in *Creative Problem Solving in School Mathematics,* and in both volumes of *Math Olympiad Contest Problems*.